# CHAPTER 7

## PROFIT ANALYSIS OF THE FIRM

Chapter Outline

I. Profit Maximization
II. Shut-Down Point
III. Break-Even Analysis
IV. Profit Maximization Versus Break-Even Analysis
V. Incremental Profit Analysis
VI. Summary: Profit Maximization and the Real World

### Questions

**1.** We mean that the goal of a firm is to make the greatest amount of profit legally possible.

**2.** There are diverse opinions among economists regarding the correct answer to this question. Other goals mentioned often include a satisfactory rate of return, growth in sales or sales maximization, and social welfare goals. However, it is probably realistic to assume that long-run profit maximization subject to a risk constraint is the predominant goal of many firms.

**3.** Break-even analysis assumes that price, average variable cost, and total fixed cost are constant. Under these assumptions the decision maker or analyst can then determine the corresponding quantity that must be sold for the firm to break even or to make a target return. Different scenarios can be developed with different prices and/or with different cost structures associated with different plants or processes. Profit maximization implicitly assumes that the total revenue function and the total cost function for the firm are known or can be estimated. That quantity of output that will maximize firm profit can then be determined by finding where marginal revenue is equal to marginal cost (subject to second order conditions). Profit maximization techniques can be used when price and average variable cost are assumed to be variable.

**4.** Incremental profit analysis is used by a firm to determine the effect on total profit that will result from a particular action, usually given that a certain set of circumstances already holds. Examples of situations in which incremental profit analysis would be useful include those in which a decision must be made regarding a special order for the firm's product, whether or not to add another product line, or (for an airline) whether or not to add another flight.

### Problems

**1.**

| *Q* | Arc *MR* | *TR* | *P* | Arc *MC* | *AFC* | *AVC* | *SAC* | *TC* |
|---|---|---|---|---|---|---|---|---|
| 0 | | 0 | $21.00 | | – | – | – | 28 |
| | 20 | | | 25 | | | | |
| 1 | | 20 | 20.00 | | 28 | 25 | 53 | 53 |
| | 18 | | | 15 | | | | |
| 2 | | 38 | 19.00 | | 14 | 20 | 34 | 68 |
| | 16 | | | 11 | | | | |
| 3 | | 54 | 18.00 | | 9.33 | 17 | 26.33 | 79 |
| | 14 | | | 5 | | | | |
| 4 | | 68 | 17.00 | | 7 | 14 | 21 | 84 |
| | 12 | | | 4 | | | | |
| 5 | | 80 | 16.00 | | 5.60 | 12 | 17.60 | 88 |
| | 8 | | | 6 | | | | |
| 6 | | 88 | 14.67 | | 4.67 | 11 | 15.67 | 94 |
| | 6 | | | 11 | | | | |
| 7 | | 94 | 13.43 | | 4 | 11 | 15 | 105 |
| | 4 | | | 19 | | | | |
| 8 | | 98 | 12.25 | | 3.50 | 12 | 15.50 | 124 |

Profit-maximizing price = $14.67 Output = 6 units

**2.** $Q_{BEP} = \dfrac{TFC}{P - AVC}$

$Q_{BEP} = \dfrac{300{,}000}{30 - 6} = \dfrac{300{,}000}{24} = \underline{\underline{12{,}500}}$ car-rental days per month

For $60,000 income before taxes:

$Q = \dfrac{300{,}000 + 60{,}000}{30 - 6} = \dfrac{360{,}000}{24} = \underline{\underline{15{,}000}}$ car-rental days per month

**3.**

| $Q$ | $P$ | $TR$ | Arc $MR$ | Arc $MC$ | $TFC$ | $AVC$ | Arc $M\pi$ |
|---|---|---|---|---|---|---|---|
| 0 | $5.00 | $ 0 | | | $60 | $ – | |
| | | | $4.90 | $3.00 | | | $ 1.90 |
| 10 | 4.90 | 49 | | | 60 | 3.00 | |
| | | | 4.70 | 1.00 | | | 3.70 |
| 20 | 4.80 | 96 | | | 60 | 2.00 | |
| | | | 3.90 | 1.00 | | | 2.90 |
| 30 | 4.50 | 135 | | | 60 | 1.67 | |
| | | | 2.50 | .50 | | | 2.00 |
| 40 | 4.00 | 160 | | | 60 | 1.38 | |
| | | | 1.50 | 1.00 | | | .50 |
| 50 | 3.50 | 175 | | | 60 | 1.30 | |
| | | | .50 | 1.50 | | | −1.00 |
| 60 | 3.00 | 180 | | | 60 | 1.33 | |
| | | | −.50 | 2.00 | | | −2.50 |
| 70 | 2.50 | 175 | | | 60 | 1.43 | |
| | | | −3.10 | 3.00 | | | −6.10 |
| 80 | 1.80 | 144 | | | 60 | 1.63 | |
| | | | −5.40 | 4.50 | | | −9.90 |
| 90 | 1.00 | 90 | | | 60 | 1.94 | |
| | | | −8.00 | 7.00 | | | −15.00 |
| 100 | .10 | 10 | | | 60 | 2.45 | |

$\pi$ is maximized at $P = \$3.50$ and $Q = 50$, since beyond that output arc $M\pi < 0$. At $Q = 50$, $\pi = TR - TC = \$175 - 60 - 50(1.30) = \underline{\underline{50}}$.

**4.** **a.** Price = $1,000

$TVC = \$30{,}000 \quad AVC = \dfrac{30{,}000}{50} = 600$

$TFC = \$10{,}000$

$Q_{BEP} = \dfrac{TFC}{P - AVC} = \dfrac{10{,}000}{1{,}000 - 600} = \dfrac{10{,}000}{400} = \underline{\underline{25}}$

**b.** $TFC = \$10{,}000 + \$5{,}000 = \$15{,}000$
Average variable selling expenses = $200

$AVC = \$300 + \$200 = \$500$

$Q_{BEP} = \dfrac{TFC}{P - AVC} = \dfrac{15{,}000}{500} = \underline{\underline{30}}$

**5.**

| $P$ | $Q$ | $TR$ | Arc $MR$ | Arc $MC$ | $AFC$ | $TVC$ | $TC$ | Arc $M\pi$ |
|---|---|---|---|---|---|---|---|---|
| $900 | 0 | $ 0 | | | $ – | $ 0 | $ 6,000 | |
| | | | $ 875 | $500 | | | | $ 375 |
| 875 | 10 | 8,750 | | | 600 | 5,000 | 11,000 | |
| | | | 825 | 400 | | | | 425 |
| 850 | 20 | 17,000 | | | 300 | 9,000 | 15,000 | |
| | | | 700 | 350 | | | | 350 |
| 800 | 30 | 24,000 | | | 200 | 12,500 | 18,500 | |
| | | | 600 | 300 | | | | 300 |
| 750 | 40 | 30,000 | | | 150 | 15,500 | 21,500 | |
| | | | 375 | 250 | | | | 125 |
| 675 | 50 | 33,750 | | | 120 | 18,000 | 24,000 | |
| | | | 225 | 200 | | | | 25 |
| 600 | 60 | 36,000 | | | 100 | 20,000 | 26,000 | |
| | | | −100 | 180 | | | | −280 |
| 500 | 70 | 35,000 | | | 85.71 | 21,800 | 27,800 | |
| | | | −300 | 200 | | | | −500 |
| 400 | 80 | 32,000 | | | 75 | 23,800 | 29,800 | |
| | | | −1,400 | 300 | | | | −1,700 |
| 200 | 90 | 18,000 | | | 66.67 | 26,800 | 32,800 | |

Profit-maximizing price = $600

Profit-maximizing output = 60 units

$T\pi = TR - TC = \$36{,}000 - \$26{,}000 = \$10{,}000$

**6.** Students' answers may vary for this problem. Our answers are only one set out of a wide variety of possible "correct" answers.

**a.** We shall assume that $3,000 of the cooks' salaries is fixed and that $1,650 of the server expense is fixed because these costs represent the minimum number of such personnel that this diner can have on hand and still be open. Actually, these costs are semivariable. We shall also assume that $150 of the food service utilities is fixed (necessary refrigeration, etc.). We shall assume that all depreciation is fixed, that $900 of the monthly advertising expense is fixed, that $600 of the transportation expense is fixed, and that all of the office salaries, supplies, and utilities are fixed—as well as the interest expenses. Our costs are thus separated as follows:

| | Variable | Fixed |
|---|---|---|
| Cooks | $ 6,000 | $ 3,000 |
| Servers | 7,850 | 1,650 |
| Food | 21,000 | 0 |
| Utilities (food service) | 750 | 150 |
| Depreciation (kitchen) | | 4,500 |
| Advertising Expense | 6,000 | 900 |
| Transportation | 900 | 600 |
| Office Salaries and Supplies | | 3,000 |
| Utilities (office) | | 600 |
| Depreciation (office) | | 600 |
| Interest Expense | | 6,000 |
| TOTAL | $42,500 | $21,000 |

$$AVC = \frac{42{,}500}{10{,}000} = \$4.25$$

$$Q_{BEP} = \frac{21{,}000}{6.00 - 4.25} = \frac{21{,}000}{1.75} = \underline{12{,}000} \text{ meals per month.}$$

**b.** These costs might include an implicit rent on the building in which the restaurant is located, implicit interest on money which the owners have invested in the restaurant, and implicit salaries for the owners' time.

**c.** Some of the restaurant's fixed costs seem high relative to total sales, such as the depreciation expense, office salaries and supplies, and interest expense. This may be an indication of considerable excess capacity. Consequently, the owners should investigate the price elasticity of demand for their meals to determine if they could increase total profit by lowering price. They might also wish to investigate the demand for alternative menu items. Unless the Crossroads Diner is newly opened, the advertising expense also seems high relative to sales volume. Thus, the cost-effectiveness of current advertising should be investigated. Also, any other areas where costs could be cut should be investigated.

**7. a.**

| Price | Quantity | Total Revenue | Marginal Revenue | Marginal Cost | *TFC* | *AVC* | Marginal Profit |
|---|---|---|---|---|---|---|---|
| $200 | 0 | $ 0 | | | $ 0 | – | |
| | | | $190 | $150 | | | $ 40 |
| 190 | 1,000 | 190,000 | | | 150,000 | $12.00 | |
| | | | 170 | 140 | | | 30 |
| 180 | 2,000 | 360,000 | | | 290,000 | 6.00 | |
| | | | 150 | 130 | | | 20 |
| 170 | 3,000 | 510,000 | | | 420,000 | 4.00 | |
| | | | 130 | 120 | | | 10 |
| 160 | 4,000 | 640,000 | | | 540,000 | 3.00 | |
| | | | 110 | 100 | | | 10 |
| 150 | 5,000 | 750,000 | | | 640,000 | 2.40 | |
| | | | 90 | 80 | | | 10 |
| 140 | 6,000 | 840,000 | | | 720,000 | 2.00 | |
| | | | 70 | 75 | | | −5 |
| 130 | 7,000 | 910,000 | | | 795,000 | 1.71 | |
| | | | 50 | 80 | | | −30 |
| 120 | 8,000 | 960,000 | | | 875,000 | 1.50 | |
| | | | 30 | 100 | | | −70 |
| 110 | 9,000 | 990,000 | | | 975,000 | 1.33 | |
| | | | 10 | 120 | | | −110 |
| 100 | 10,000 | 1,000,000 | | | 1,095,000 | 1.20 | |
| | | | −10 | 140 | | | −150 |
| 90 | 11,000 | 990,000 | | | 1,235,000 | 1.09 | |

**b.** $P = \$140$, $Q = 6{,}000$; closest to where $MR = MC$ and marginal profit is not negative.

**8. a.**

| | Variable Costs | Fixed Costs |
|---|---|---|
| Direct Labor | $ 700,000 | |
| Direct Materials | 350,000 | |
| Variable Overhead | 150,000 | |
| Fixed Overhead | | $ 600,000 |
| Commissions | 500,000 | |
| Travel | 500,000 | 100,000 |
| Advertising Expense | 250,000 | 50,000 |
| Office Supplies | | 10,000 |
| Office Salaries | 50,000 | 40,000 |
| Interest Expense | | 500,000 |
| TOTAL | $2,500,000 | $1,300,000 |

$$AVC = \frac{TVC}{Q} = \frac{\$2{,}500{,}000}{1{,}000{,}000} = \$2.50$$

$$Q_{BEP} = \frac{TFC}{P - AVC} = \frac{1{,}300{,}000}{5 - 2.50} = \frac{1{,}300{,}000}{2.50}$$

$$= \underline{\underline{520{,}000}} \text{ bags per year}$$

**b.** *Yes*. Presently, average direct labor cost $= \dfrac{700{,}000}{1{,}000{,}000} = \$.70$ per bag.

If this figure drops to \$.15, *AVC* will decrease by \$.55 to \$1.95. *TFC* would rise by \$900,000 to \$2,200,000. If the firm sells 2,000,000 bags at a price of \$4.50, its total profit would be:

| | | |
|---|---|---|
| Total revenue | = 2,000,000 bags at \$4.50 = | \$9,000,000 |
| Less: | | |
| Total variable cost | = 2,000,000 bags at \$1.95 = | 3,900,000 |
| | | 5,100,000 |
| Less: | | |
| Total fixed cost | = | 2,200,000 |
| Total income before taxes | | \$2,900,000 |

Net income would rise from \$1,200,000 to \$2,900,000.

**9. a.**

| | Variable Costs | Fixed Costs |
|---|---|---|
| Direct Materials | \$195,000 | |
| Direct Labor | 210,000 | |
| Fixed Manufacturing Expenses | | \$50,000 |
| Delivery Expenses | 30,000 | |
| Sales Commissions | 50,000 | |
| Advertising Expense | 10,000 | |
| Travel Expense | 5,000 | |
| Fixed Administrative and Selling Expenses | | 10,000 |
| TOTAL | \$500,000 | \$60,000 |

$$AVC = \frac{TVC}{Q} = \frac{500{,}000}{100{,}000} = \$5$$

$$Q_{BEP} = \frac{TFC}{P - AVC} = \frac{60{,}000}{7-5} = \frac{60{,}000}{2}$$

$$= \underline{\underline{30{,}000}} \text{ cases per month}$$

**b.** *Yes*. For foreign offer:

| | | |
|---|---|---|
| Need | 40,000 | cases per month for 3 months. |
| Less: | | |
| From inventory | 10,000 | cases per month for 3 months. |
| From excess capacity | 20,000 | cases per month for 3 months. |
| From domestic sales | 10,000 | cases per month for 3 months. |

Variable costs not connected with this sale:

| | |
|---|---|
| Delivery | \$30,000 |
| Sales Commissions | 50,000 |
| Advertising | 10,000 |
| Travel | 5,000 |
| | \$95,000 |

$$\text{Reduction in } AVC = \frac{95{,}000}{100{,}000} = \$.95$$

Therefore, $AVC = \$4.05$ for the foreign offer.

| | |
|---|---|
| Profit contribution from foreign offer: | |
| (120,000) × (5.75 − 4.05) = | \$204,000 |
| Less: | |
| Profit contribution lost from domestic sale: | |
| (30,000) × (7.00 − 5.00) = | 60,000 |
| Net increase in profit contribution | \$144,000 |

**10. a.** To max profit, $MR - MC = 0$. With the fixed government price, $MR = 80$ dinars. Thus, $80 - 20 - .0002Q = 0$; $60 = .0002Q$; $Q = \underline{\underline{300{,}000}}$ cans per month.

**b.** $TR = 80(300{,}000) = 24{,}000{,}000$ dinars.
$STC = 8{,}000{,}000 + 6{,}000{,}000 + 9{,}000{,}000 = 23{,}000{,}000$ dinars.
Profit $= 24{,}000{,}000 - 23{,}000{,}000 = \underline{\underline{1{,}000{,}000}}$ dinars.

**11. a.** To max profit, $MR - MC = 0$.
$340 - 5Q - 40 + 10Q - Q^2 = 0$
$-Q^2 + 5Q + 300 = 0$; $(-Q + 20)(Q + 15) = 0$
$Q = 20$
$P = 340 - 2.5(20) = \underline{\underline{290}}$

**b.** $TR = 20(290) = 5{,}800$
$STC = 3{,}000 + 800 - 2{,}000 + 2{,}666.67 = 4{,}466.67$
Profit $= 5{,}800 - 4{,}466.67 = \underline{\underline{1{,}333.33}}$

**12. a.** $Q_b = TFC/(P - AVC)$; $Q_b = 1960/2.80 = \underline{\underline{700}}$

**b.** $Q = (\text{profit} + TFC)/(P - AVC)$; $Q = (12{,}000 + 1{,}960)/2.80 = \underline{\underline{4{,}986}}$

**13.** Setting $MR = SMC$ one obtains $370 - 2Q = 10 + 2Q$, so $4Q = 360$ and $Q = \underline{\underline{90}}$.
From the $AR$ equation, $AR = P = 370 - 90 = \underline{\underline{\$280}}$.
Profit $= TR - STC = \$280(90) - 10{,}500 - 10Q - Q^2 = \$25{,}200 - 19{,}500 = \underline{\underline{\$5{,}700}}$.

**C1. a.**

$$MR = \frac{dTR}{dQ} = 21 - 2Q = 0$$
$$2Q = 21$$
$$\underline{\underline{Q = 10.5}}$$
$$\left(\frac{d^2TR}{dQ^2} = -2 < 0, \text{ so } TR \text{ is maximized at } Q = 10.5.\right)$$

**b.**

$$T\pi = TR - TC = 21Q - Q^2 - (1/3)Q^3 + 3Q^2 - 9Q - 6$$
$$\frac{dT\pi}{dQ} = 21 - 2Q - Q^2 + 6Q - 9 = 0$$
$$-Q^2 + 4Q + 12 = 0$$
$$Q^2 - 4Q - 12 = 0$$
$$(Q - 6)(Q + 2) = 0$$
$$Q - 6 = 0 \qquad Q + 2 = 0$$
$$\underline{\underline{Q = 6}} \qquad \underline{\underline{Q = -2}}$$
$$\left(\frac{d^2T\pi}{dQ^2} = -2Q + 4. \text{ At } Q = 6, \frac{d^2T\pi}{dQ^2} = -8, \text{ so } T\pi \text{ is maximized at } Q = 6.\right)$$

**c.** Total profit $= -(1/3)Q^3 + 2Q^2 + 12Q - 6$
$= -(1/3)(6)^3 + 2(6)^2 + 12(6) - 6$
$= -(1/3)(216) + 2(36) + 72 - 6$
$= -72 + 72 + 66$
$= \underline{\underline{\$66}}.$

**C2.** $T\pi = TR - TC = 50Q - Q^2 - 100 + 4Q - 2Q^2$

$$\frac{d^2T\pi}{dQ} = 50 - 2Q + 4 - 4Q = 0$$
$$-6Q = -54$$
$$\underline{\underline{Q = 9}}$$
$$\left(\frac{d^2T\pi}{dQ^2} = -6 < 0, \text{ so } T\pi \text{ is maximized at} Q = 9.\right)$$

$T\pi = -3Q^2 + 54Q - 100$
$= -3(9)^2 + 54(9) = 100$
$= -3(81) + 486 - 100$
$= -243 + 386$
$= \underline{\underline{143}}.$

**C3. a.** $Q = 220 - P$
$-P = Q - 220$
$P = -Q + 220$
$TR = (-Q + 220)Q = -Q^2 + 220Q$

**b.** $T\pi = -Q^2 + 220Q - 1{,}000 - 80Q + 3Q^2 - (1/3)Q^3$
$= -1{,}000 + 140Q + 2Q^2 - (1/3)Q^3$

$$\frac{dT\pi}{dQ} = 140 + 4Q - Q^2 = 0$$
$$(Q + 10)(-Q + 14) = 0$$

$Q = -10$ $\quad$ $\underline{\underline{Q = 14}}$
Not possible $\quad$ $P = -Q + 220 = \underline{\underline{\$206}}$

**c.** $T\pi = -1{,}000 + 140(14) + 2(14)^2 - (1/3)(14)^3$
$= -1{,}000 + 1{,}960 + 392 - 914.67$
$= \underline{\underline{\$437.33}}$

**C4. a.** $AFC = 4850/25 = \underline{\underline{194}}$

**b.** $SMC = 40 - 3Q + 0.12Q^2$; $dSMC/dQ = -3 + 0.24Q = 0$; $Q = \underline{\underline{12.5}}$

**c.** $AVC = 40 - 1.5Q + 0.04Q^2$; $dAVC/dQ = -1.5 + 0.08Q = 0$; $Q = \underline{\underline{18.75}}$

**d.** To max profit, $MR - MC = 0$. Since $P = MR = 190$, $MR - MC = 190 - 40 + 3Q - 0.12Q^2 = 0$
$-12Q^2 + 3Q + 150 = 0$; dividing by 0.12, $-Q^2 + 25Q + 1250 = 0$;
$(-Q + 50)(Q + 25) = 0$; $Q = \underline{\underline{50}}$
Profit $= 190(50) - 4850 - 2000 + 3750 - 5000 = \underline{\underline{1400}}.$

**C5.** From the given demand curve, $P = 1400 - 4Q$ and $MR = 1400 - 8Q$. From the given cost function, $SMC = 200 - 18Q + Q^2$
To max profit, $MR - MC = 0$; $1400 - 8Q - 200 + 18Q - Q^2 = 0$
$-Q^2 + 10Q + 1200 = 0$; $(-Q + 40)(Q + 30) = 0$; $Q = 40$
$P = 1400 - 4(40) = \underline{\underline{1240}}$
Profit $= 1240(40) - 20{,}000 - 8{,}000 + 14{,}400 - 21{,}333.33 = \underline{\underline{14{,}666.67}}.$

**C6. a.** $MR = 150 - 4Q$.

**b.** i. From the cost function, $SMC = 30 - 6Q + Q^2$. Therefore, setting marginal profit equal to zero:

$$M\pi = MR - SMC = 150 - 4Q - 30 + 6Q - Q^2 = 0$$
$$-Q^2 + 2Q + 120 = 0; (-Q + 12)(Q + 10) = 0; Q = \underline{\underline{12}}.$$

ii. $P = 150 - 2Q = 150 - 24 = \underline{\underline{126}}$.

iii. $T\pi = 126(12) - 500 - 30(12) + 3(144) - (1/3)(1{,}728) = \underline{\underline{508}}$.

**C7.** The demand function equation is $Q_c = 50 - 2P_c + 0.1F + 0.002I - 0.01K$. The given values for the independent variables other than $P_c$ are:

$F = 2{,}000$
$I = \$90{,}000$
$K = 15{,}000$

When these are substituted into the demand function, they yield
$Q_c = 50 - 2P_c + 200 + 180 = 280 - 2P_c$.
Thus, $P_c = 140 - 0.5Q_c$, and $MR = 140 - Q_c$.
From the $TVC$ function, $SMC = 20 + 3Q$. Setting this equal to $MR$ yields $4Q = 120$, and $Q = \underline{\underline{30}}$. By substitution, $P = 140 - 15 = \underline{\underline{\$125}}$. Therefore, the total profit contribution will be

$$T\pi_c = \$125(30) - 20(30) - 1.5(30)^2$$
$$= \$3{,}750 - 600 - 1{,}350 = \underline{\underline{1{,}800}}.$$

**C8. a.**

$$Q_S = 20{,}000 = 100P_S \qquad Q_B = 50{,}000 - 400P_B$$
$$-100P_S = Q_S - 20{,}000 \qquad -400P_B = Q_B - 50{,}000$$
$$P_S = -.01Q_S + 200 \qquad P_B = -.0025Q_B + 125$$

$STC = 100{,}000 + 25(Q_S + Q_B)$
Constraint: $Q_S + Q_B = 17{,}500$

$$LT\pi = (-.01Q_S + 200)Q_S + (-.0025Q_B + 125)Q_B - 100{,}000$$
$$-25(Q_S + Q_B) - \lambda(Q_S + Q_B - 17{,}500)$$
$$LT\pi = .01Q_S^2 + 175Q_S - .0025Q_B^2 + 100Q_B - 100{,}000$$
$$-\lambda(Q_S + Q_B - 17{,}500)$$

(1) $\dfrac{\partial LT\pi}{\partial Q_S} = -.02Q_S + 175 - \lambda = 0$

(2) $\dfrac{\partial LT\pi}{\partial Q_S} = -.005Q_B + 100 - \lambda = 0$

(3) $\dfrac{\partial LT\pi}{\partial \lambda} = -Q_S - Q_B + 17{,}500 = 0$

From (1) and (2):

$$-.02Q_S + 175 - \lambda = 0$$
$$.005Q_B - 100 + \lambda = 0$$
$$\overline{-.02Q_S + .005Q_B + 75 = 0}$$
$$-2Q_S + .5Q_B + 7{,}500 = 0$$
$$2Q_S + 2.0Q_B - 35{,}000 = 0 \quad \text{From (3)}$$
$$\overline{2.5Q_B - 27{,}500 = 0}$$
$$2.5Q_B = 27{,}500$$
$$Q_B = \underline{\underline{11{,}000}}$$
$$P_B = -.0025(11{,}000) + 125$$
$$= -27.50 + 125$$
$$= \underline{\underline{97.50}}$$

$Q_S + 11{,}000 = 17{,}500$
$Q_S = \underline{\underline{6{,}500}}$
$P_S = -.01(6{,}500) + 200$
$= -65 + 200$
$= \underline{\underline{135}}$

**b.** From (1):
$-.02(6{,}500) + 175 - \lambda = 0$
$-130 + 175 - \lambda = 0$
$\underline{\underline{\lambda = 45}}$

If capacity were to increase by 1 unit, total profit would increase by approximately $45.

**C9. a.** Maximize $T\pi = 52C - .06C^2 + 70I - .1I^2 + .01CI - 8{,}000$, subject to the capacity constraint, $I + 1.5C = 500$. Forming the Lagrangian and setting its partial derivatives equal to zero:

$H = 52C - .06C^2 + 70I - .1I^2 + .01CI - 8{,}000 + \lambda(500 - 1.5C - I)$
$\partial H/\partial C = 52 - .12C + .01I - 1.5\lambda = 0$
$\partial H/\partial I = 70 + .01C - .2I - \lambda = 0$
$\partial H/\partial \lambda = 500 - 1.5C - I = 0$
Multiply $\partial H/\partial I$ by $-1.5$ and add it to $\partial H/\partial C$ to obtain:

$-53 - .135C + .31I = 0.$

Multiply the constraint ($\partial H/\partial \lambda$) by .31 and add it to the above to obtain:

$102. - .6C = 0$; $C = \underline{\underline{170}}$.

From the constraint, $500 - 1.5(170) - I = 0$; $I = \underline{\underline{245}}$.

**b.** Substituting the solution values of $C$ and $I$ into the objective function:
$T\pi = 8{,}840 - 1{,}734 + 17{,}150 - 6{,}002.5 + 416.5 + 8{,}000 = \underline{\underline{10{,}670}}$.

**c.** From $\partial H/\partial I$, $\lambda = 70 + .01(170) - .2(245) = 22.7$. $\lambda$ is the rate of change of profit with respect to the constraint, so increasing capacity would apparently increase profit. However, an increase in capacity would change the profit function, especially the constant term, 8,000, which may represent fixed cost. While Stan has good reason to consider increasing capacity, one would have to know how such a move would alter the objective function in order to properly analyze the issue.

# APPENDIX 7
# Linear Programming and the Firm

## Questions

**1.** Linear programming techniques can deal with optimization problems involving linear functions and inequality constraints. Calculus techniques can deal with optimization problems involving linear or nonlinear functions and equality constraints. Thus, linear programming techniques are more useful if constraints are in the form of inequalities but all functions are linear, whereas calculus techniques are more useful if the functions are nonlinear but all constraints are in the form of equalities. (It should be noted that nonlinear programming techniques can handle both nonlinear functions and inequality constraints.)

**2.** It could be useful with decisions such as the determination of the product combination that would maximize profit, given certain technology and input constraints, or the determination of the advertising combination that would minimize cost while meeting various coverage constraints. Other examples can be found in Chapter 8 and the corresponding problems.

**3.** Total profit contribution is the difference between total revenue and total variable cost, and total fixed cost is fixed; therefore, when total profit contribution is at a maximum, total profit is also at a maximum.

**4.** One gets information about the opportunity costs associated with the decision variables and the constraints at the optimal point. See Problems 2, 3, 4, and 5 for examples.

**5.** The dual may have fewer decision variables than does the primal program. By analyzing the opportunity cost values at the dual program solution point, one can immediately locate the optimal solution for the primal program. Again, see Problems 2, 3, and 4 for examples.

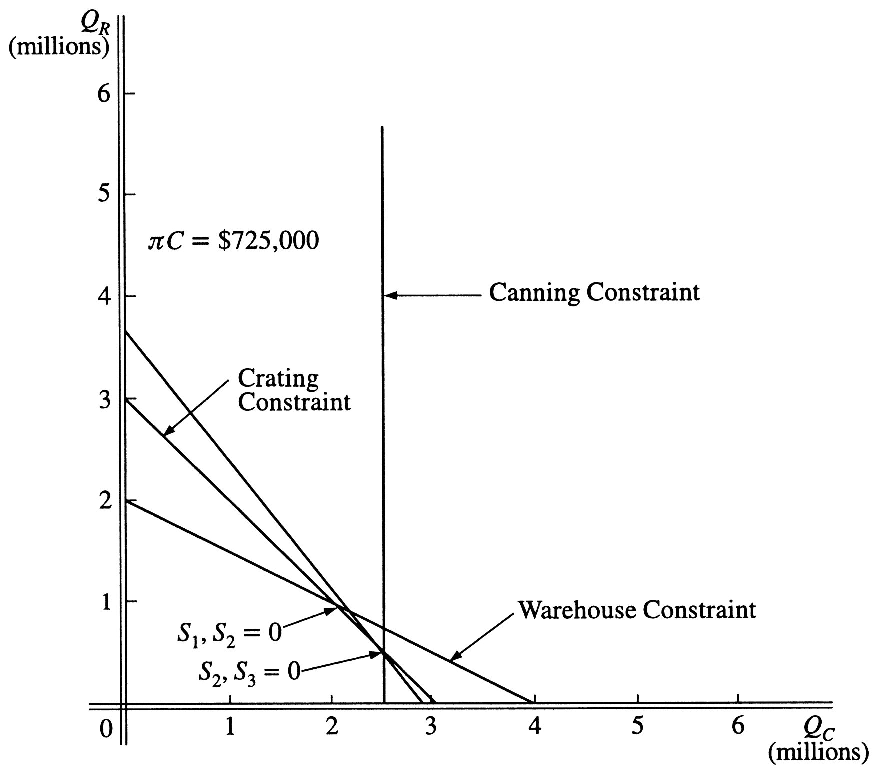
## Problems

**1.** $Q_R$ = quantity of raw pineapples
$Q_C$ = number of cans of pineapples
$\pi C = \$.20Q_R + \$.25Q_C$

Constraints:
$.6Q_R + 3Q_C \leq 1{,}200{,}000$ warehouse
$.2Q_R + .2Q_C \leq 600{,}000$ crating
$.1Q_C \leq 250{,}000$ canning

Constraints with slack variables:
$.6Q_R + .3Q_C + S_1 = 1{,}200{,}000$
$.2Q_R + .2Q_C + S_2 = 600{,}000$
$.1Q_C + S_2 = 250{,}000$

By algebraically checking all of the boundary points of the feasible region, we can determine that the point defined by $S_2$ and $S_3 = 0$ is the optimal point. (See the graphical solution to determine the boundary points of the feasible region.)



**a.** If $S_2, S_3 = 0$:

$$.6Q_R + .3Q_C + S_1 = 1{,}200{,}000$$
$$.2Q_R + .2Q_C = 600{,}000$$
$$.1Q_C = 250{,}000$$
$$\text{Therefore, } Q_C = \underline{\underline{2{,}500{,}000}} \text{ cans}$$

$$.2Q_R + .2(2{,}500{,}000) = 600{,}000$$
$$.2Q_R = 100{,}000$$
$$Q_R = \underline{\underline{500{,}000}} \text{ raw pineapples}$$

**b.** $\pi C = \$.20(500{,}000) + \$.25(2{,}500{,}000) = \underline{\underline{\$725{,}000}}$

Also, $.6Q_R + .3Q_C + S_1 = 1{,}200{,}000$

$.6(500{,}000) + .3(2{,}500{,}000) + S_1, = 1{,}200{,}000$

$S_1 = \underline{\underline{150{,}000}}$ units

For an example of another boundary solution (but one which is nonoptimal), we check the point defined by $S_1$, $S_2 = 0$

(1) $.6Q_R + .3Q_C = 1{,}200{,}000$

(2) $\underline{.2Q_R + .2Q_C = \quad 600{,}000}$

$.6Q_R + .3Q_C = 1{,}200{,}000$

Multiply (2) by $-3/2$ $\quad \underline{-.3Q_R - 3Q_C = -900{,}000}$

$.3Q_R \qquad = 300{,}000$

$Q_R \qquad = \underline{\underline{1{,}000{,}000}}$ pineapples

$.2Q_C = 600{,}000 - 200{,}000 = 400{,}000$

$Q_C = \underline{\underline{2{,}000{,}000}}$ cans

$\pi C = \$.20(1{,}000{,}000) + \$.25(2{,}000{,}000) = \$700{,}000$

$.1Q_C + S_3 = 250{,}000$

$200{,}000 + S_3 = 250{,}000$

$S_3 = \underline{\underline{50{,}000}}$ units

In a similar manner, the remaining boundary points may be checked.

**2.** $Q_M$ = quantity of motor homes

$Q_T$ = quantity of travel trailers

$\pi C = \$4{,}000Q_M + \$3{,}0000Q_T$

Constraints:

$2.0Q_M \leq 300$ power train assembly

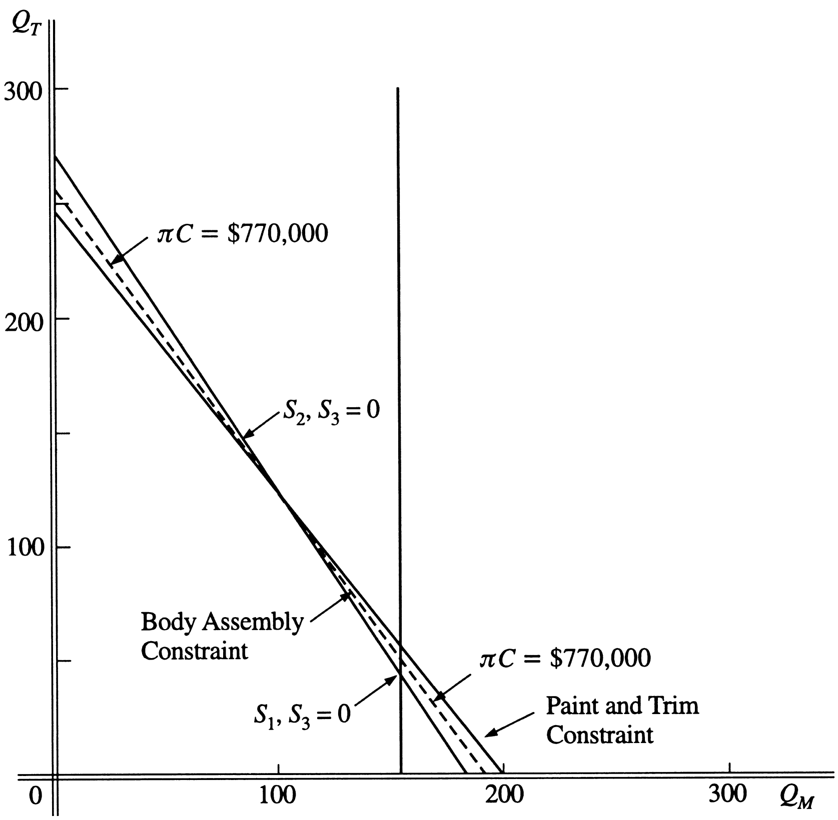
$2.5Q_M + 2.0Q_T \leq 500$ paint and trim

$3.0Q_M + 2.0Q_T \leq 540$ body assembly

Constraints with slack variables:

$$2.0Q_M \qquad + S_1 = 300$$
$$2.5Q_M + 2.0Q_T + S_2 = 500$$
$$3.0Q_M + 2.0Q_T + S_3 = 540$$

By algebraically checking all of the boundary points of the feasible region, we can determine that the point defined by $S_2$ and $S_3 = 0$ is the optimal point. (See the graphical solution to determine the boundary points of the feasible region.)




**a.** If $S_2, S_3 = 0$

$$\begin{aligned} 2.5Q_M + 2.0Q_T &= 500 \\ 3.0Q_M + 2.0Q_T &= 540 \\ \hline -.5Q_M &= -40 \\ Q_M &= \underline{\underline{80}} \text{ motor homes} \\ 3.0(80) + 2.0Q_T &= 540 \\ Q_T &= \underline{\underline{150}} \text{ travel trailers} \end{aligned}$$

**b.** $\pi C = \$4{,}000(80) + \$3{,}000(150) = \underline{\underline{\$770{,}000}}$

$$\begin{aligned} 2.0(80) + S_1 &= 300 \\ S_1 &= \underline{\underline{140}} \text{ hours} \end{aligned}$$

For an example of another boundary solution (but one which is nonoptimal), we check the point defined by $S_1$, $S_3 = 0$.

$$\begin{aligned} \text{Therefore, } 2.0Q_M &= 300 \\ Q_M &= \underline{\underline{150}} \text{ motor homes} \\ 3.0(150) + 2.0Q_T &= 540 \\ 2.0Q_T &= 90 \\ Q_T &= \underline{\underline{45}} \text{ travel trailers} \\ \pi C = \$4{,}000(150) + \$3{,}000(45) &= \underline{\underline{735{,}000}} \\ 2.5(150) + 2.0(45) + S_2 &= 500 \\ 375 + 90 + S_2 &= 500 \\ S_2 &= \underline{\underline{35}} \text{ hours} \end{aligned}$$

In a similar manner, the remaining boundary points may be checked.

**3.** Minimize $C = 300V_1 + 500V_2 + 540V_3$

Subject to: $2.0V_1 + 2.5V_2 + 3.0V_3 \geq 4{,}000$

$2.0V_2 + 2.0V_3 \geq 3{,}000$

Also, $2.0V_1 + 2.5V_2 + 3.0V_3 - L_M = 4{,}000$

$2.0V_2 + 2.0V_3 - L_T = 3{,}000$

**a.** The additional useful information that Holiday on Wheels will obtain from solving the dual program is the marginal opportunity cost of using the three fixed inputs: the power train assembly, the paint and trim capacity, and the body assembly. These values are given by $V_1$, $V_2$, and $V_3$, respectively, and represent (approximately) the additional profit contribution that would be obtained if the respective constraints could be relaxed by one unit.

**b.** From our solution to the primal program (Problem 2), we know that the optimal dual program solution will occur where $V_1$ $LM$, and $L_T = 0$. Thus,

(1) $2.5V_2 + 3.0V_3 = 4{,}000$

(2) $\underline{2.0V_2 + 2.0V_3 = 3{,}000}$

$2.5V_2 + 3.0V_3 = 4{,}000$

Multiply (2) by −3/2 $\underline{-3.0V_2 - 3.0V_3 = -4{,}500}$

$-.5V_2 = -500$

$V_2 = \underline{\underline{1{,}000}}$

$2(1{,}000) + 2V_3 = 3{,}000$

$2V_3 = 1{,}000$

$V_3 = \underline{\underline{500}}$

$C = 300(0) + 500(\$1{,}000) + 540(\$500) = \underline{\underline{\$770{,}000}}$

**c.**

$V_1 = 0$ implies that the marginal opportunity cost of one more hour of power train assembly capacity is zero. This result was obtained because the firm has excess power train assembly capacity.

$V_2 = \$1{,}000$ indicates the (approximate) increase in profit contribution that would be obtained if there were one more hour of paint and trim capacity available.

$V_3 = \$500$ indicates the (approximate) increase in profit contribution that would be obtained if there were one more hour of body assembly capacity available.

$L_M$ and $L_T = 0$ indicates that the opportunity cost of the resources utilized in producing motor homes and travel trailers, respectively, is just equal to the profit contribution from their respective production.

$C = \$770{,}000$ indicates that the total opportunity cost assigned to the fixed inputs is equal to the maximum profit contribution obtainable from them.

**4.** $Q_G$ = tons of grain

$Q_S$ = tons of silage

$C = \$80Q_G + \$30Q_S$

Constraints:

$200Q_G + 40Q_S \geq 1{,}200$ protein

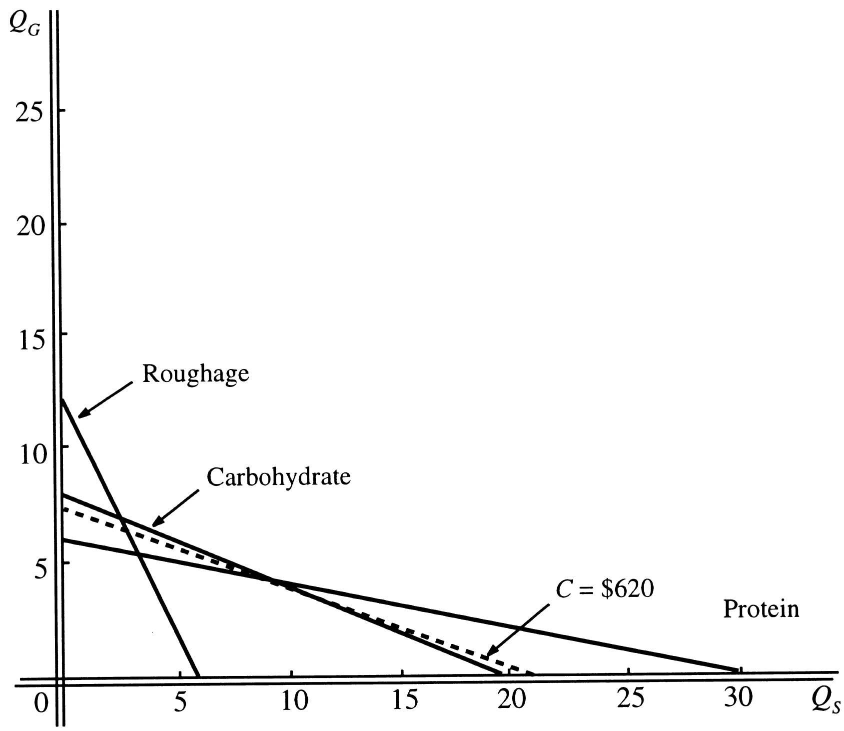
$1{,}000Q_G + 400Q_S \geq 8{,}000$ carbohydrates

$300Q_G + 600Q_S \geq 3{,}600$ roughage

Constraints with slack variables:

$$200Q_G + 40Q_S - S_1 = 1{,}200$$
$$1{,}000Q_G + 400Q_S - S_2 = 8{,}000$$
$$300Q_G + 600Q_S - S_3 = 3{,}600$$

By algebraically checking all of the boundary points of the feasible region, we can determine that the point defined by $S_1$ and $S_2 = 0$ will give the least-cost combination of feed. (See the graphical solution to determine the boundary points of the feasible region.)




**a.** If $S_1$ and $S_2 = 0$,

$$(1) \quad 200Q_G + 40Q_S = 1{,}200$$
$$(2) \quad \underline{1.000Q_G + 400Q_S = 8{,}000}$$
$$200Q_G + 40Q_S = 1{,}200$$
$$\text{Divide (2) by } (-10) \quad \underline{-100Q_G - 40Q_S = -800}$$
$$100Q_G = 400$$
$$Q_G = \underline{\underline{4}} \text{ tons of grain}$$

$$200(4) + 40Q_G = 1{,}200$$
$$40Q_S = 400$$
$$Q_S = \underline{\underline{10}} \text{ tons of silage}$$
$$300(4) + 600(10) - S_3 = 3{,}600$$
$$-S_3 = -3{,}600$$
$$S_3 = 3{,}600$$
$$C = \$80(4) + \$30(10) = \underline{\underline{\$620}}$$

For an example of another boundary solution (but one which is not least cost), we check the point defined by $S_2, S_3 = 0$.

$$\begin{aligned} &(1) \quad 1{,}000Q_G + 400Q_S = 8{,}000 \\ &(2) \quad 300Q_G + 600Q_S = 3{,}600 \end{aligned}$$

Divide (1) by 2 $\quad 500Q_G + 200Q_S = 4{,}000$

Divide (2) by $-3$ $\quad -100Q_G - 200Q_S = -1{,}200$

$$400Q_G = 2{,}800$$

$$Q_G = \underline{\underline{7}} \text{ tons of grain}$$

$$1{,}000(7) + 400Q_S = 8{,}000$$

$$400Q_S = 1{,}000$$

$$Q_S = \underline{\underline{2.5}} \text{ tons of silage}$$

$$200(7) + 40(2.5) - S_1 = 1{,}200$$

$$1{,}400 + 100 - S_1 = 1{,}200$$

$$-S_1 = -300$$

$$S_1 = 300$$

$$C = \$80(7) + \$30(2.5) = \underline{\underline{635}}$$

In a similar manner, the remaining boundary points may be checked.

**b.** $C = \$80(4) + \$30(10) = \underline{\underline{\$620}}$.

**c.** Dual Program

Maximize $V = 1{,}200V_1 + 8{,}000V_2 + 3{,}600V_3$

Subject to: $200V_1 + 1{,}000V_2 + 300V_3 + L_G = 80$

$$40V_1 + 400V_2 + 600V_3 + L_S = 30$$

From the primal program solution, we know that $L_G$, $L_S$, and $V_3$ are equal to zero at the optimal point. Thus,

$$(1)\ 200V_1 + 1{,}000V_2 = 80$$

$$40V_1 + 400V_2 = 30$$

Divide (1) by $-5$ $\quad -40V_1 - 200V_2 = -16$

$$40V_1 + 400V_2 = 30$$

$$200V_2 = 14$$

$$V_2 = \underline{\underline{\$.07}}$$

$$200V_1 + 1{,}000(.07) = 80$$

$$200V_1 = 10$$

$$V_1 = \underline{\underline{\$.05}}$$

$$V = 1{,}200(\$.05) + 8.000(\$.07) + 3{,}600(0) = \underline{\underline{\$620}}$$

The marginal cost of the protein constraint is \$.05, and the marginal cost of the carbohydrate constraint is \$.07. The marginal cost of the roughage constraint is zero.

**d.** This information would be useful to BFB if it were trying to determine whether or not to alter the protein, carbohydrate, and roughage constraints because $V_1$, $V_2$, and $V_3$ indicate the marginal costs, respectively, of doing so. These marginal costs can be compared with the marginal benefits resulting from changing the constraints.

# INTEGRATING CASE 2A
# Frontier Concrete Products Company

## Questions

**1.** Cost of one unit of capital per hour $= \dfrac{18{,}000}{260 \times 8} = 8.65$, approximately.

| | Q = 30 cubic yards per hour | | | |
|---|---|---|---|---|
| | **Plant 1** | **Plant 2** | **Plant 3** | **Plant 4** |
| | $K = 4$, $L = 1$ | $K = 3$, $L = 2$ | $K = 2$, $L = 5$ | $K = 1$, $L = 10$ |
| Labor cost/hr. | $ 7.00 | $ 14.00 | $ 35.00 | $ 70.00 |
| Capital cost/hr. | 34.60 | 25.95 | 17.30 | 8.65 |
| Subtotal | $ 41.60 | $ 39.95 | $ 52.30 | $ 78.65 |
| Delivery, materials | 445.20 | 445.20 | 445.20 | 445.20 |
| TOTAL | $486.80 | $485.15 | $497.50 | $523.85 |

| | Q = 45 cubic yards per hour | | | |
|---|---|---|---|---|
| | **Plant 5** | **Plant 6** | **Plant 7** | **Plant 8** |
| | $K = 6$, $L = 11.5$ | $K = 4.5$, $L = 3$ | $K = 3$, $L = 7.5$ | $K = 1.5$, $L = 15$ |
| Labor cost/hr. | $ 10.50 | $ 21.00 | $ 52.50 | $105.00 |
| Capital cost/hr. | 51.90 | 38.92 | 25.95 | 12.98 |
| Subtotal | $ 62.40 | $ 59.92 | $ 78.45 | $117.98 |
| Delivery, materials | 667.80 | 667.80 | 667.80 | 667.80 |
| TOTAL | $730.20 | $727.72 | $746.25 | $785.78 |

| | Q = 60 cubic yards per hour | | | |
|---|---|---|---|---|
| | **Plant 9** | **Plant 10** | **Plant 11** | **Plant 12** |
| | $K = 8$, $L = 2$ | $K = 6$, $L = 4$ | $K = 4$, $L = 10$ | $K = 2$, $L = 20$ |
| Labor cost/hr. | $ 14.00 | $ 28.00 | $ 70.00 | $ 140.00 |
| Capital cost/hr. | 69.20 | 51.90 | 34.60 | 17.30 |
| Subtotal | $ 83.20 | $ 79.90 | $104.60 | $ 157.30 |
| Delivery, materials | 890.40 | 890.40 | 890.40 | 890.40 |
| TOTAL | $973.60 | $970.30 | $995.00 | $1,047.70 |

*Constant* returns to scale.

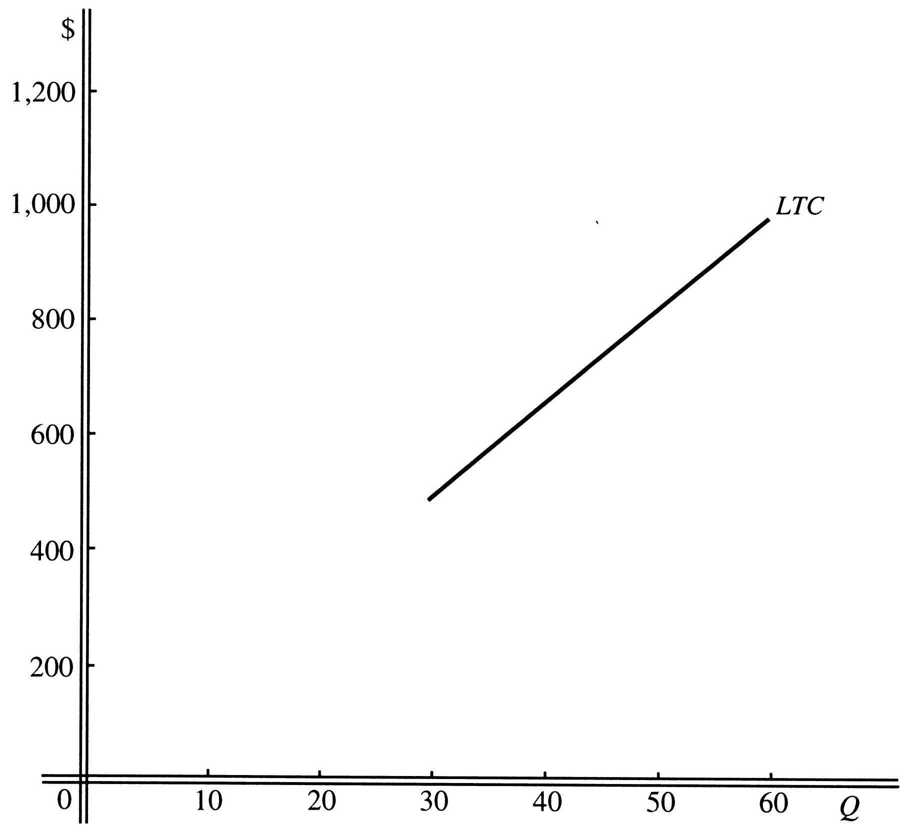
**2.** See table for Question 1.

**3.** For $Q = 30$: $K = 3$ and $L = 2$.

For $Q = 45$: $K = 4.5$ and $L = 3$.

For $Q = 60$: $K = 6$ and $L = 4$.

**4.**




**5.**

| *Q* | *LTC* | *LAC* | Arc *LMC* |
|---|---|---|---|
| 0 | $ 0 | $ – | |
| 30 | $485.15 | 16.17 | $16.17 |
| 45 | 727.72 | 16.17 | 16.17 |
| 60 | 970.30 | 16.17 | 16.17 |

**6.**

| *Q* | *Price* | *TR* | Arc *MR* |
|---|---|---|---|
| 0 | $ – | $ 0 | |
| 20 | 24.00 | 480.00 | $24.00 |
| 30 | 22.34 | 670.20 | 19.02 |
| 40 | 21.26 | 850.40 | 18.02 |
| 50 | 20.40 | 1,020.00 | 16.96 |
| 60 | 19.66 | 1,179.60 | 15.96 |
| 70 | 19.00 | 1,330.00 | 15.04 |
| 80 | 18.38 | 1,470.40 | 14.04 |

**7.** $TFC$ = \$96,000 per year

Variable costs per yard:

Capital costs: \$12,000 per year if produce to capacity

$12,000 ÷ (260 × 8 × 60) = $ .10 per yard

Labor costs: \$28.00/60 yds. = .47 per yard

Raw materials, delivery expenses: \$14.84 per yard

\$15.41 per yard

$$Q_{BEP} = \frac{96{,}000}{20 - 15.41} = 20{,}915$$ cubic yards, approximately, per year or 10 yards per hour.

**8.** 60 cubic yards per hour.

**9.** 50 cubic yards per hour.

**10.** Opportunity costs for any money or tangible assets which Frontier's owners invested in the firm should be computed. Also, the opportunity costs for any time which they spend on the firm's business should be included.

# INTEGRATING CASE 2B
# Shanghai Magnificent Harmony Foundry I

This case can be solved *a la* breakeven analysis, but the overiding concern is not the break-even point. Rather the Committee wants to know whether the amount of production dedicated to the project will result in a profitable undertaking, given its desire to cover the allocated fixed costs.

**1.** "*Best case*." The given variable cost data sum to \$0.32 per lb. Since Fei knows that freight will cost \$.10 per lb., he can land the covers in the U.S. for \$0.42 per lb. Thus his profit contribution per cover will be \$0.06(160) = \$9.60. (The 160 is the average weight of a cover in pounds.) The total number of 160−lb. covers that can be obtained from 6,500 short tons of casting output is 13,000,000/160 = 81,250.

Accordingly,

$T\pi_c$ = \$9.60(81,250) = \$780,000.

Allocated fixed cost = 2,500,000 *RY*/8 = \$312,500.

Gross profit = \$780,000 − 312,500 = \$467,500.

Happy Mr. Fei!

**2.** The tariff of \$.02 per lb., if absorbed by SMHF, would reduce its f.o.b. price to \$0.46 per lb. and its unit profit contribution to \$.04 per lb. or \$6.40 per cover. Still, with a quantity sold of 81,250 covers,

$T\pi_c$ = \$6.40(81,250) = \$520,000,

which again would more than cover the allocated fixed cost of \$312,500.

**3.** In the worst case, the result is further deteriorated by an increase in direct materials to \$0.12 per lb., which will drop unit profit contribution to \$0.02 per lb. Thus,

$T\pi_c$ = \$3.20(81,250) = \$260,000,

which will not cover the allocated fixed cost of \$312,500.