PART 2

PRODUCTION, COST, AND PROFIT MAXIMIZATION

CHAPTER 5

PRODUCTION ANALYSIS

Chapter Outline

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Chapter Summary

Appendix 5: Mathematics of Determining the Least Cost Combination of Inputs

Questions

1. A production function is a mathematical statement of the way that the quantity of output of a particular product depends on the use of specific inputs or resources by the firm. A total product function indicates the maximum amount of output that can be produced using different amounts of one variable input and a fixed amount of other inputs. Thus, the total product function applies in the situation in which only one input can vary; the production function applies when all inputs are variable.

2. For a least cost combination of inputs to be achieved, it must be the case that $\frac{MP_1}{P_1} = \frac{MP_2}{P_2} = \cdots = \frac{MP_n}{P_n}$, where subscripts 1, 2, . . ., n represent inputs. The firm cannot always attain a least cost combination of inputs in the short run because some inputs are fixed.

3. No, because the least cost combination of inputs for any given level of output merely ensures that level of output is produced as cheaply as possible. Revenue must also be considered when the firm determines the profit–maximizing level of output.

4. The "law" of diminishing marginal productivity states that if equal increments of one variable input are added (holding all other inputs fixed), after some point, the marginal product of the variable input will fall. If the firm has decreasing returns to scale, a given percentage increase in all inputs will result in a smaller percentage increase in the firm's output.

5. An isoquant curve indicates all of the different input combinations that will enable a firm to produce a particular level of output. An isocost curve indicates all the different input combinations that can be utilized for a given level of cost. These two curves are useful to the firm because they help it to determine least cost combinations of inputs by giving it information on the rate at which inputs can be substituted production-wise relative to the rate at which they can be substituted cost-wise.

6. Ridge lines bound the economic region of production, which is that region of production on the isoquant map where the marginal products of both inputs are positive or zero. The firm would not wish to operate using combinations of inputs that are outside this region of production because the marginal product of an input would be negative.

7. The expansion path is a line connecting all least cost combinations of inputs (on the isoquant–isocost map) for a given input price ratio. The expansion path would likely change if the ratio of input prices changed (that is, if the slope of the isocost curve changed). Also, a change in technology that caused a change in the substitutability of inputs could cause the expansion path to change.

Problems

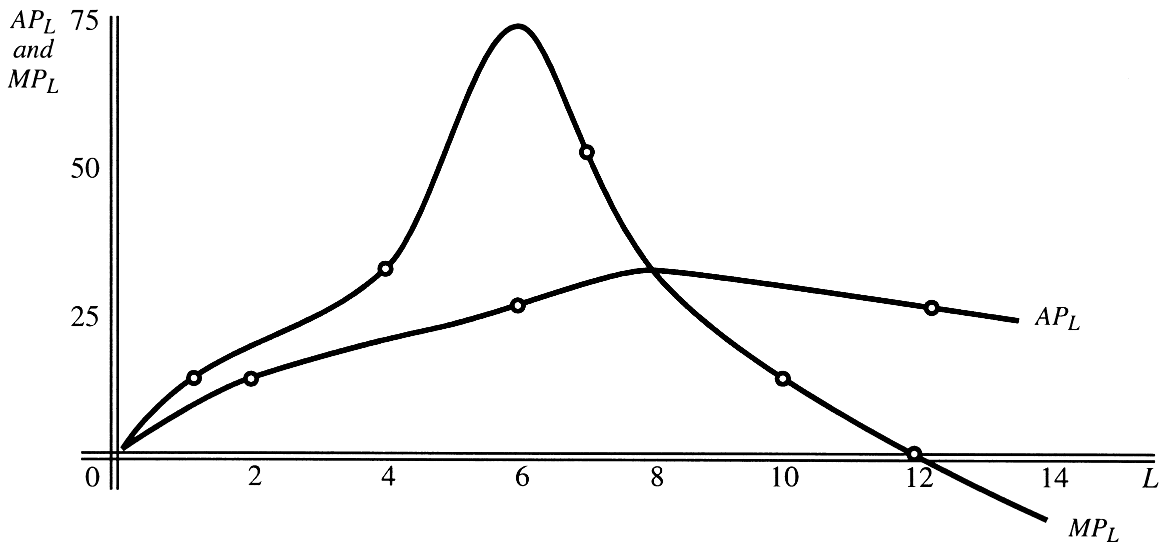
1. **a.** For artists, $MP_L/P_L = 96/480 = 0.2$.
For robots, $MP_K/P_K = 210/840 = 0.25$.

The robots are more productive per dollar spent and probably should be bought.

b. $96/P_L = 0.25$; $0.25P_L = 96$; $P_L = 96/0.25 = \underline{\underline{\$384}}$.

c. Does she expect any change in the wage rate of artists or in the operating costs of robots? Either or both will affect her decision.

2. **a.** The completed AP_L and MP_L diagram follows.



Note: For the combinations of L and Q that were labeled in the given total product curve diagram, the values of APL and MPL are set out in the following table.

L	Q	AP_L	Arc MP_L
0	0	–	
2	25	12.50	12.50
6	150	25.00	31.25
8	250	31.25	50.00
12	300	25.00	12.50

b. Max of AP_L occurs where ray from origin is tangent to TP_L, or at $L = 8$. $AP_L = 250/8 = \underline{\underline{31.25}}$.

c. A tangent drawn to the inflection point at $L = 6$ will form a right triangle with the L-axis and the perpendicular at $L = 6$ that has a height of 150 and a base of about 2. Thus, the maximum MP_L is about $150/2 = \underline{\underline{75}}$.

3. For the least cost combination of inputs,

$$\frac{MP_L}{P_L} = \frac{MP_K}{P_K}$$

$$\frac{5}{6} < \frac{10}{10} = 1,$$

so this combination of inputs is not optimal.

More output per additional dollar spent is obtained from capital than from labor. Thus, over the long run, Diamond should substitute capital for labor until $\frac{MP_L}{P_L} = \frac{MP_K}{P_K}$.

4. **a.** decreasing **d.** decreasing
b. increasing **e.** increasing
c. constant **f.** increasing

5. For a least cost combination of inputs,

$$\frac{MP_L}{P_L} = \frac{MP_K}{P_K}$$

$$\frac{4}{7} < \frac{40}{30}$$

$(.57 < 1.33)$,

so the firm would get more output per additional dollar spent if it used the machine.
Purchase the machine.
Additional data to consider might include the following:

- Future wage rate levels
- How the cost per day associated with using the machine was figured, such as how many days per year it was assumed that the machine would be used to pick apples
- How many days per year the company can expect to use the machine in the future

6. **a.** The completed table follows:

MP_L	L	Q	AP_L
	0	0	–
4.0			
	5	20	4.0
8.0			
	10	60	6.0
6.0			
	15	90	6.0
4.0			
	20	110	5.5
3.0			
	25	125	5.0
2.0			
	30	135	4.5
1.0			
	35	140	4.0

b. Since we know MP_K, P_K, and P_L, all that is necessary is to select from the table the MP_L for the appropriate range of output (90 to 110 units). Thus $MP_L/P_L = 4/40 = 0.1$ and $MP_K/P_K = 24/120 = 0.2$, showing that capital's marginal product per dollar is twice that of labor. The firm should certainly consider employing more capital and less labor.

7. **a.** Constant returns to scale; doubling the inputs results in a doubling of the level of output.

b.

$L = 3$		
K	AP_K	MP_K
1	87.00	
		35
2	61.00	
		28
3	50.00	
		23
4	43.25	
		21
5	38.80	
		19
6	35.50	

c.

$K = 1$		
L	AP_L	MP_L
1	50.00	
		21
2	35.50	
		16
3	29.00	
		13
4	25.00	
		12
5	22.40	
		10
6	20.33	

d. For a least cost combination of inputs,

$$\frac{MP_L}{P_L} = \frac{MP_K}{P_K}$$

$MP_L = 13$, $MP_K = 35$

$\frac{13}{10} < \frac{35}{20}$, or $(1.3 < 1.75)$.

This is not a least cost combination of inputs because output per additional dollar spent is greater for capital.

8. For a least cost combination of inputs,

$$\frac{MP_L}{P_L} = \frac{MP_K}{P_K}$$

$$\frac{2}{9} < \frac{10}{30}$$

$(.22 < .33)$

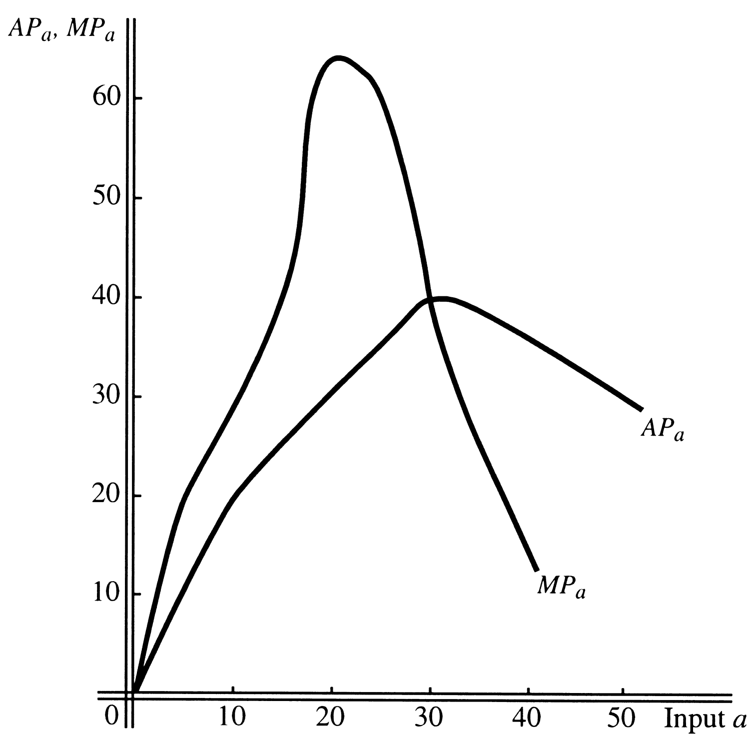
a. Rent the backhoe, as the contractor would get more dirt moved per dollar spent by using it.

b. TC using only labor $= 300 \times \$9.00 = \$2{,}700$. TC using the backhoe would require $\frac{600}{10} = 60$ hours or 8 days, since the backhoe must be rented for full 8-hour days. Cost $= \$30 \times 64$ hours $= \$1{,}920$. The cost of the project is less with the backhoe.

9. **a.** and **b.**

Input a	Ap_a	MP_a
0	–	
10	20	20
20	30	40
30	40	60
50	30	15

c.



d. 30 units and 20 units of input a, respectively.

e. Immediately after input a = 20 units.

10. a. C

b. *B; A*

c. $Y; I_3$

d. *R; Q*

e. *F; J; R*

11. Cost per hour of capital $= \dfrac{\$12{,}000}{240 \times 8} = \$6.25$

$$\frac{MP_K}{P_K} = \frac{40}{\$6.25} = 6.4 \text{ units per } \$1$$

$$\frac{MP_L}{P_L} = \frac{60}{\$11.50} = 5.2 \text{ units per } \$1$$

Robot, because additional output per additional $1 is greater for the robot.

12. a.

Units of L	TP_L	AP_L	Arc MP_L
1	40	40	
			40
2	80	40	
			40
3	120	40	
			40
4	160	40	
			40
5	200	40	
			40
6	240	40	

b. $\dfrac{MP_K}{P_K} = \dfrac{30}{\$30} = 1.0$ units of output per $1

$\dfrac{MP_L}{P_L} = \dfrac{40}{\$10} = 4.0$ units of output per $1

No, the additional output per additional $1 is greater for labor than for capital.

c. 2 units of capital, 6 units of labor, $TC = \$120$.

d. Increasing; if all inputs are increased by a certain percentage, output will increase by a greater percentage.

e. Increases; capital and labor are imperfect substitutes, so more capital increases the productivity of labor.

13. This is related to the chapter discussion of marginal revenue product and optimal use of a single variable input, a subject that will be discussed in more detail in Chapter 12. Nonetheless, there is enough information in Chapter 5 to provide answers. Given that the price of output (MR in this case) is $8, that materials cost ($MC_m$) is $3 per unit of output, that $MP_L = 20 - 0.2L$, and that the marginal cost of labor is constant at $12, we have:

a. The capacity of plant occurs where $MP_L = 0 = 20 - 0.2L$. Thus, rearranging terms, $L = 20/0.2 = \underline{\underline{100}}$.

b. Given the data, net marginal revenue $= NMR = (MR - MC_m) = (8 - 3) = \$5$, and $MRP_L = NMR(MP_L) = 5(20 - 0.2L) = 100 - L$. Setting $MRP_L = MC_L$ yields $100 - L = 12$, and $L = \underline{\underline{88}}$.

C1. **a.** $Q = 4KL + 3L^2 - (1/3)L^3$, so
$4KL + 3L^2 - (1/3)L^3 = 100$ is the isoquant equation for $Q = 100$.

b.
$$dQ = 4L\,dK + (4K + 6L - L^2)dL = 0$$
$$-4L\,dK = (4K + 6L - L^2)dL$$
$$\frac{dK}{dL} = \frac{4K + 6L - L^2}{-4L}$$

This expression can also be derived using

$$\frac{dK}{dL} = -\frac{MP_L}{MP_K}$$
$$MP_L = \frac{\partial Q}{\partial L} = 4K + 6L - L^2,$$
$$MP_K = \frac{\partial Q}{\partial K} = 4L, \text{ so}$$
$$\frac{dK}{dL} = \frac{-(4K + 6L - L^2)}{4L}$$

c. From (b), $MP_L = 4K + 6L - L^2$.
When $K = 5$, $MP_L = 20 + 6L - L^2$.

d. Diminishing returns set in when MP_L reaches a maximum.

$$\frac{dMP_L}{dL} = 6 - 2L = 0$$
$$-2L = -6$$
$L = 3$, when diminishing returns set in.

C2. **a.** $AP_L = Q/L = 400 - 0.5L$

b. $MP_L = dQ/dL = 400 - L$

c. Where $MP_L = 0$; or at $L = \underline{\underline{400}}$.

d. Occurs where $MP_L = 0$ or $Q = 400(400) - 0.5(400)^2$.
Thus $Q = 160{,}000 - 80{,}000 = \underline{\underline{80{,}000}}$.

C3. If $K = 4$, then
$Q = 48L + 4L^2 - (1/3)L^3$

a. At capacity,

$$\frac{dQ}{dL} = MP_L = 48 + 8L - L^2 = 0$$
$$(-L + 12)(L + 4) = 0$$
$$L = 12$$
$$Q = 48(12) + 4(144) - (1/3)(1{,}728)$$
$$= 576 + 576 - 576 = \underline{\underline{576}}.$$

b.
$$AP_L = 48 + 4L - (1/3)L^2$$
$$\frac{dAP_L}{dL} = 4 - (2/3)L = 0$$
$$\underline{\underline{L = 6}}.$$

c. Where MP_L is *maximum*,

$$\frac{dMP_L}{dL} = 8 - 2L = 0$$

$\underline{\underline{L = 4}}$

$Q = 48(4) + 4(16) - (1/3)(64) = 192 + 64 - 21.33 = 234.67.$

C4. **a.** $MP_Z = 160 + 36Z - Z^2 = 0$; $(-Z + 40)(Z + 4) = 0$

$\underline{\underline{Z = 40}}$.

b. At $Z = 40$, $Q = 6{,}400 + 28{,}800 - 21{,}333.33 = \underline{\underline{13{,}866.67}}$.

c. $dMP_Z/dZ = 36 - 2Z = 0$; $Z = 18$.

$MP_Z = 160 + 648 - 324 = \underline{\underline{484}}$

d. $AP_Z = 160 + 18Z - Z^2/3$

$dAP_Z/dZ = 18 - 2Z/3 = 0$

$2Z = 54$; $Z = \underline{\underline{27}}$.

C5. **a.** Differentiate to obtain MP_L, then set $MP_L = 0$. Put the resulting value for Q into the total product function to obtain maximum output.

$MP_L = dQ/dL = 44 + 20L - L^2 = 0$

$(-L + 22)(L + 2) = 0$; the positive root is $L = 22$.

$Q = 44(22) + 10(22)^2 - (1/3)(22)^3 = 968 + 4{,}840 - 3{,}549.33 = \underline{\underline{2{,}258.67}}$.

b. Set the derivative of MP_L equal to zero.

$dMP_L/dL = 20 - 2L = 0$; $L = \underline{\underline{10}}$.

c. The level of output where MP_L is maximum or where $L = 10$. Put $L = 10$ into the total product function to obtain the corresponding output.

$Q = 44(10) + 10(10)^2 - (1/3)(10)^3 = 440 + 1{,}000 - 333.33 = \underline{\underline{1{,}106.67}}$.

d. Define the AP_L equation, then set its derivative equal to zero to obtain the relevant L. Put the L value into the AP_L equation to determine maximum APL.

$AP_L = Q/L = 44 + 10L - (1/3)L^2$

$dAP_L/dL = 10 - (2/3)L = 0$; $L = \underline{\underline{15}}$. $AP_L = 44 + 150 - 75 = \underline{\underline{119}}$.

C6. **a.** Set $MP_L = 0$. $MP_L = 84 + 22L - 2L^2 = 0$.

$(-2L + 28)(L + 3) = 0$; $L = \underline{\underline{14}}$.

b. $Q = 84(14) + 11(14)^2 - (2/3)(14)^3 = 1{,}176 + 2{,}156 - 1{,}829.33$

$= \underline{\underline{1{,}502.67}}$.

c. $AP_L = Q/L = 84 + 11L - (2/3)L^2$

$dAP_L/dL = 11 - (4/3)L = 0$; $L = \underline{\underline{8.25}}$.

d. $dMP_L/dL = 22 - 4L = 0$; $L = 5.5$; $MP_L = 84 + 121 - 60.5 = \underline{\underline{144.5}}$.

C7. Like Problem 13, this is related to the chapter discussion of marginal revenue product and optimal use of a single variable input, a subject that will be discussed in more detail in Chapter 12. Nontheless, there is enough information in Chapter 5 to provide answers. Given that the price of output (MR in this case) is \$20, that materials cost (MC_m) is \$7 per unit of output, that from the total product function one obtains $MP_L = 50 - 0.2L$, and that the marginal cost of labor is constant at \$130, we have:

a. The capacity of plant occurs where $MP_L = 0 = 50 - 0.2L$. Thus, rearranging terms, $L = 50/0.2 = 250$. Substituting this L value into TP_L yields $Q = 50(250) - 0.1(250)^2 = 12{,}500 - 6{,}250 = \underline{\underline{6{,}250}}$.

b. Given the data, net marginal revenue = $NMR = (MR - MC_m) = (20 - 7) = \$13$ and $MRP_L = NMR(MP_L) = 13(50 - 0.2L) = 650 - 2.6L$. Setting $MRP_L = MC_L$ yields $650 - 2.6L = 130$, and $L = \underline{\underline{200}}$. When this L value is substituted into the total product function, one obtains $Q = 50(200) - 0.1\,(200)^2 = 10{,}000 - 4{,}000 = \underline{\underline{6{,}000}}$.

C8. Given the total product function $Q = 84Z - .01Z^2$, we obtain the following results.

a. $MP_Z = dQ/dZ = 84 - .02Z$

b. This MP function is a straight line with a negative slope. Therefore, MP_Z always falls.

c. $AP_Z = Q/Z = 84 - .01Z$

d. Where $AP_Z = 0$, $0 = 84 - .01Z$, and $Z = 8{,}400$. If you substitute $Z = 8{,}400$ into the Q function, $Q = 0$. That is because we are at the point where the total product curve, which reaches its maximum when Z = 4,200, has fallen from the maximum to again cross the Z axis. This total product curve is shaped like the total revenue curve for a straight-line demand curve.

APPENDIX 5
Mathematics of Determining the Least Cost Combination of Inputs

Problem

1. Maximize:

$Q = L^2 + 10LK + K^2 - \lambda(5L + 20K - 1{,}150)$

(1) $\dfrac{\partial Q}{\partial L} = 2L + 10K - 5\lambda = 0$

(2) $\dfrac{\partial Q}{\partial K} = 10L + 2K - 20\lambda = 0$

(3) $\dfrac{\partial Q}{\partial \lambda} = -(5L + 20K - 1{,}150) = 0$

Multiply (1) by −4

Add:

(4) $-8L - 40K + 20\lambda = 0$

(2) $\underline{10L + 2K - 20\lambda = 0}$

(5) $2L - 38K = 0$

Divide (5) by 2:

(6) $L - 19K = 0$

Divide (3) by 5:

(7) $-L - 4K = -230$

Add:

(6) $L - 19K = 0$

(7) $\underline{-L - 4K = -230}$

(8) $-23K = -230$

Thus, $K = -230/-23 = \underline{\underline{10}}$

and from (6) $L - 19K = L - 190 = 0$, so $L = \underline{\underline{190}}$

Then using (1):

$2(190) + 10(10) - 5\lambda = 480 - 5\lambda = 0$,

so $\lambda = \underline{\underline{96}}$.

$Q = L^2 + 10LK + K^2 = (190)^2 + 10(190)(10) + (10)^2$

$= 36{,}100 + 19{,}000 + 100 = \underline{\underline{55{,}200}}$ units

2. Form the Lagrangian as follows:

$H = 40X^{.5}\,Y^{.5} + \lambda(16{,}000 - 100X - 20Y)$.

Set the partials with respect to *X, Y*, and λ equal to zero.

$\partial H/\partial X = 20X^{-.5}\,Y^{.5} - 100\lambda = 0$
$\partial H/\partial Y = 20X^{.5}\,Y^{-.5} - 20\lambda = 0$
$\partial H/\partial \lambda = 16{,}000 - 100X - 20Y = 0$.

Multiply the second equation above by -5 and add it to the first.

$-100X^{.5}\,Y^{-.5} + 100\lambda = 0$

$\underline{20X^{-.5}\,Y^{.5} - 100\lambda = 0}$

$-100X^{.5}\,Y^{-.5} + 20X^{-.5}\,Y^{.5} = 0$; $20Y^{.5}/X^{.5} = 100X^{.5}/Y^{.5}$; $20Y = 100X$ and $100X - 20Y = 0$.

Adding the last expression to $\partial H/\partial \lambda$,

$16{,}000 - 100X - 20Y + 100X - 20Y = 0$. $40Y = 16{,}000$; $Y = \underline{\underline{400}}$. By substitution, $X = \underline{\underline{80}}$.

From the objective function, $Q = 40(80)^{.5}\,(400)^{.5} = 40(8.94)(20) = \underline{\underline{7.152}}$.

Note: The second order condition for a maximum is that the Hessian determinant for the set of three partials above be positive. In this case it does have a positive value, as one would expect for a Cobb-Douglas production function.